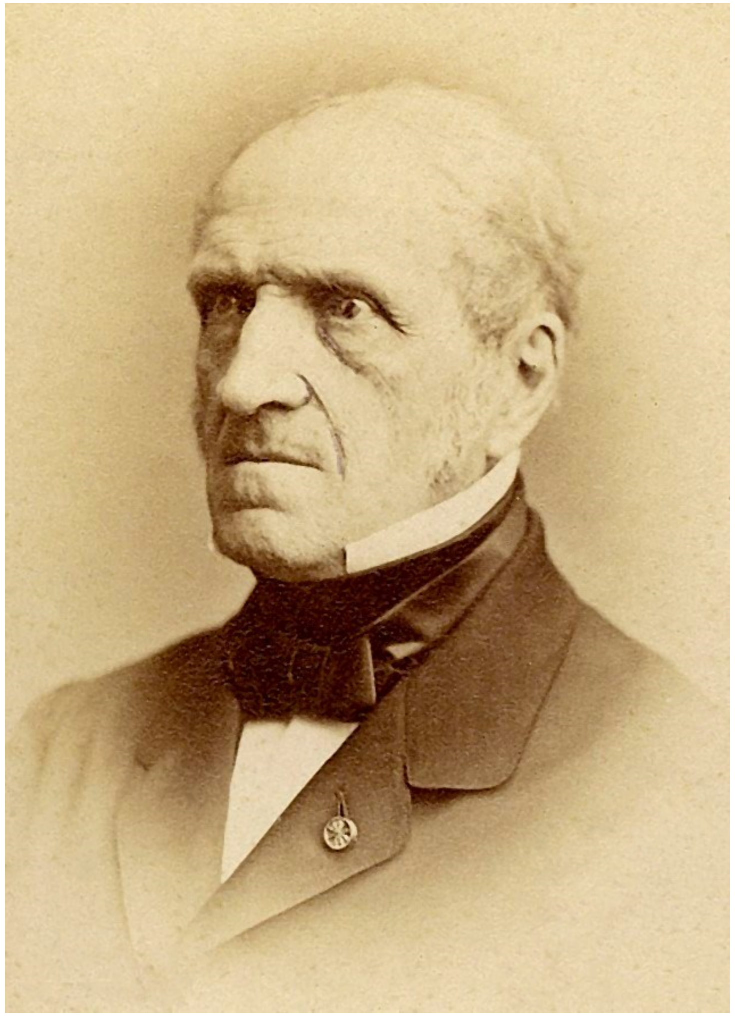
TWO POSITIONS OF A RIGID BODY

(planer motion, only)

Michel Chasles (1793-1880)



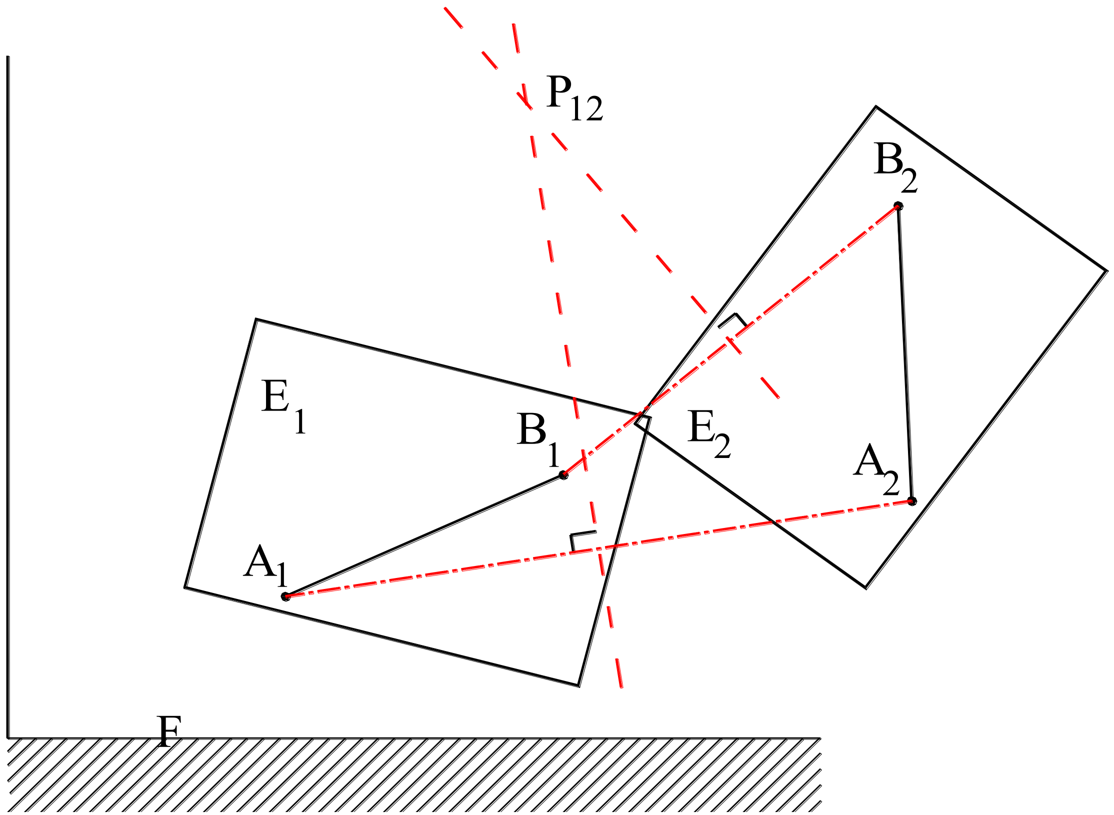
Chasles' Theorem

«The translation of a rigid body in a plane motion occurs most simply by a rotation about the pole P_{12} which is located at the point of intersection of the perpendicular bisectors of two pairs of homologous points.«

Homologous Points: The different locations of a point on the moving frame at different positions of the moving frame projected to the fixed frame

A, and B are any two points on the moving body

A_1, A_2 – Homologous points.



The position of a rigid body is defined when:

a) The coordinates of a point on the rigid body and the orientation of a line relatieve to a fixed reference are known.

b) The coordinates of any two points on the rigid body are known.

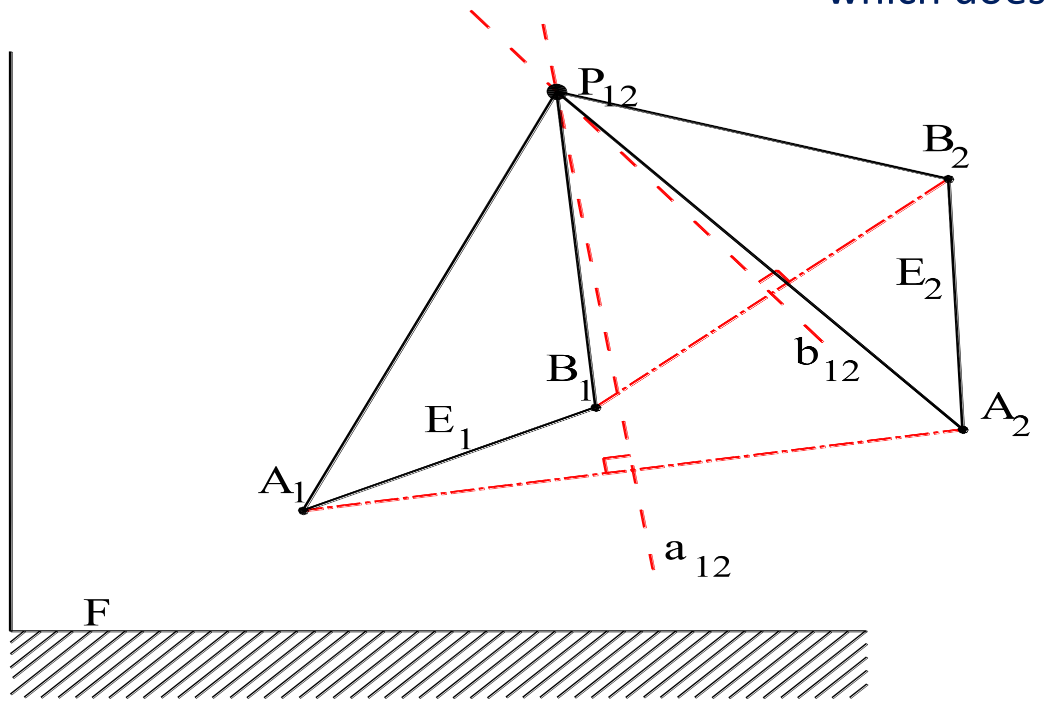
Proof of Chasles' Theorem:

P_{12} is common for positions 1 and 2 (i.e. İt is the only point which does not move)

$A_1B_1 = A_2B_2 = AB$

$A_1P_{12} = A_2P_{12}$

$B_1P_{12} = B_2P_{12}$



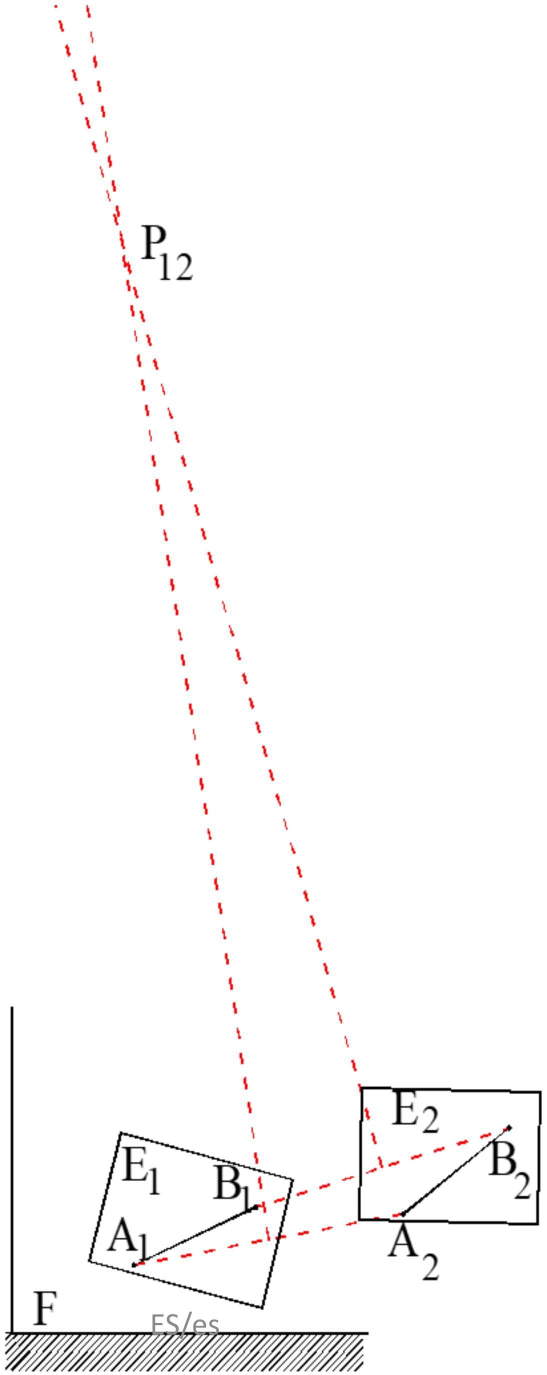
$\triangle A_1B_1P_{12} = \triangle A_2B_2P_{12}$

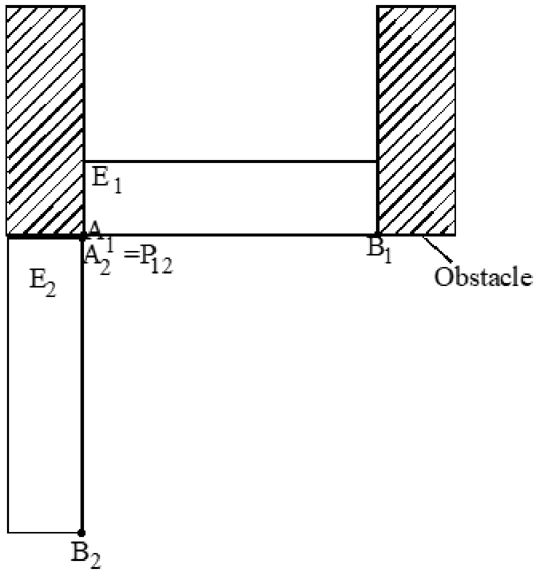
Side_Side-Side

This is the simplest solution to move a rigid body in plane from one position to another.

However, a rotation about the pole may not be feasible.

1. Whe the pole is to far away from the rigid body.
2. Done can not reach the pole physically.
3. There is an obstacle during this rotary motion
4. …





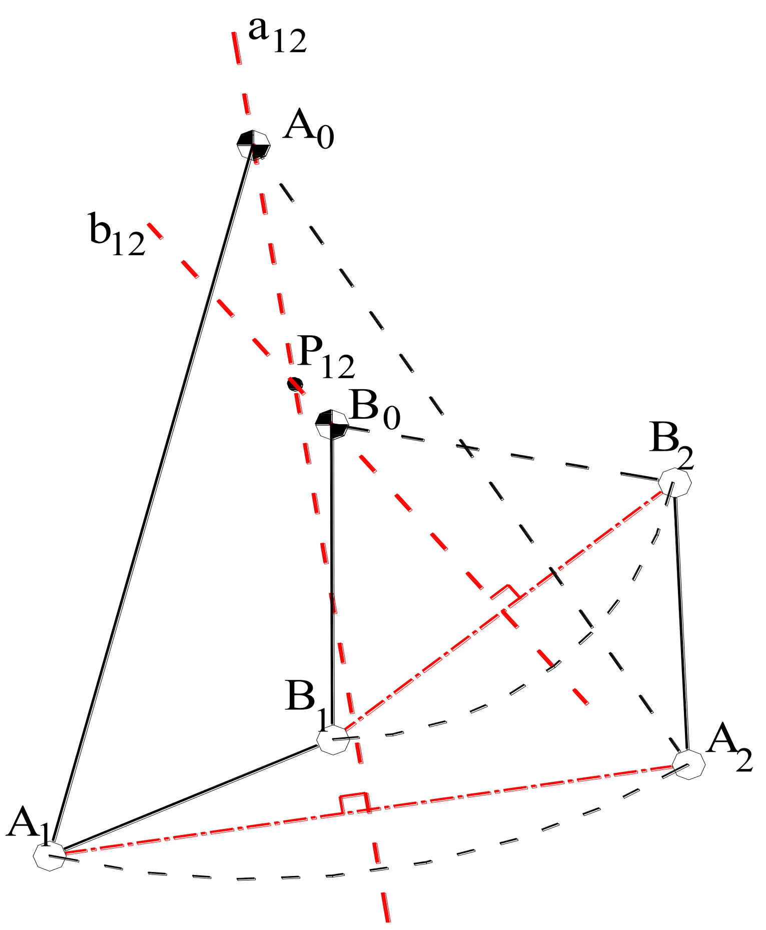
Four-Bar Mechanism for two positions:

1. Select A_0 at any point on the perpendicular bisector to A_1A_2 (a_{12}).
2. Select B_0 at any point on the perpendicular bisector to B_1B_2 (b_{12}).

a) **Point A is any point on the rigid body. You need two parameters to determine the location of point A.**

b) **A_0 is any point on the perpendicular bisector to A_1 A_2 (a_{12}). You need one parameter.**

Therefore, you have 6 parameters that can be selected.

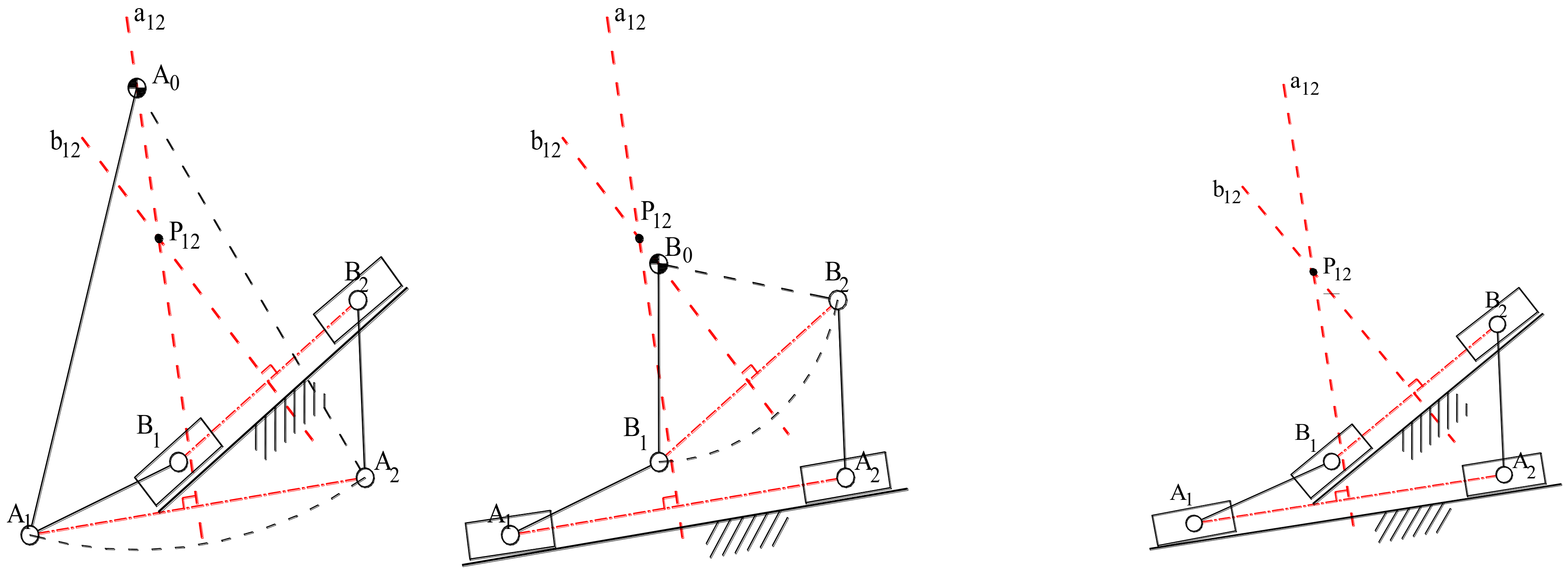


A: "Circle Point", "Moving Pivot"

A_0: "Center Point", "Fixed Pivot"

You must CHECK resulting mechanism.
No Guarantie is given for the motion in between the positions.

Slider-Crank or Double Slider Mechanism Solutions:



Select A_0 and/or B_0 at infinity. In such a case the path of point A and/or B will be a straight line, which can be realized by attaching a slider whose axis is A_1A_2 and/or B_1B_2.

You must <u>CHECK</u> resulting mechanism.
No Guarantie is given for the motion in between the positions.

Some important relations:

$\alpha = \angle A_1P_{12}B_1 = \angle A_2P_{12}B_2$

$\phi_{12} = \angle A_1P_{12}A_2 = \angle B_1P_{12}B_2$

$\angle A_1P_{12}B_2 = \angle A_1P_{12}A_2 + \angle A_2P_{12}B_2 = \phi_{12} + \alpha$

and:

$\angle A_1P_{12}B_2 = \angle A_1P_{12}N + \angle N P_{12}M + \angle MP_{12}B_2$

Since points M and N are on the perpendicular bisector:

$\angle A_1P_{12}N = ½ \angle A_1P_{12}A_2 = ½ \phi12$

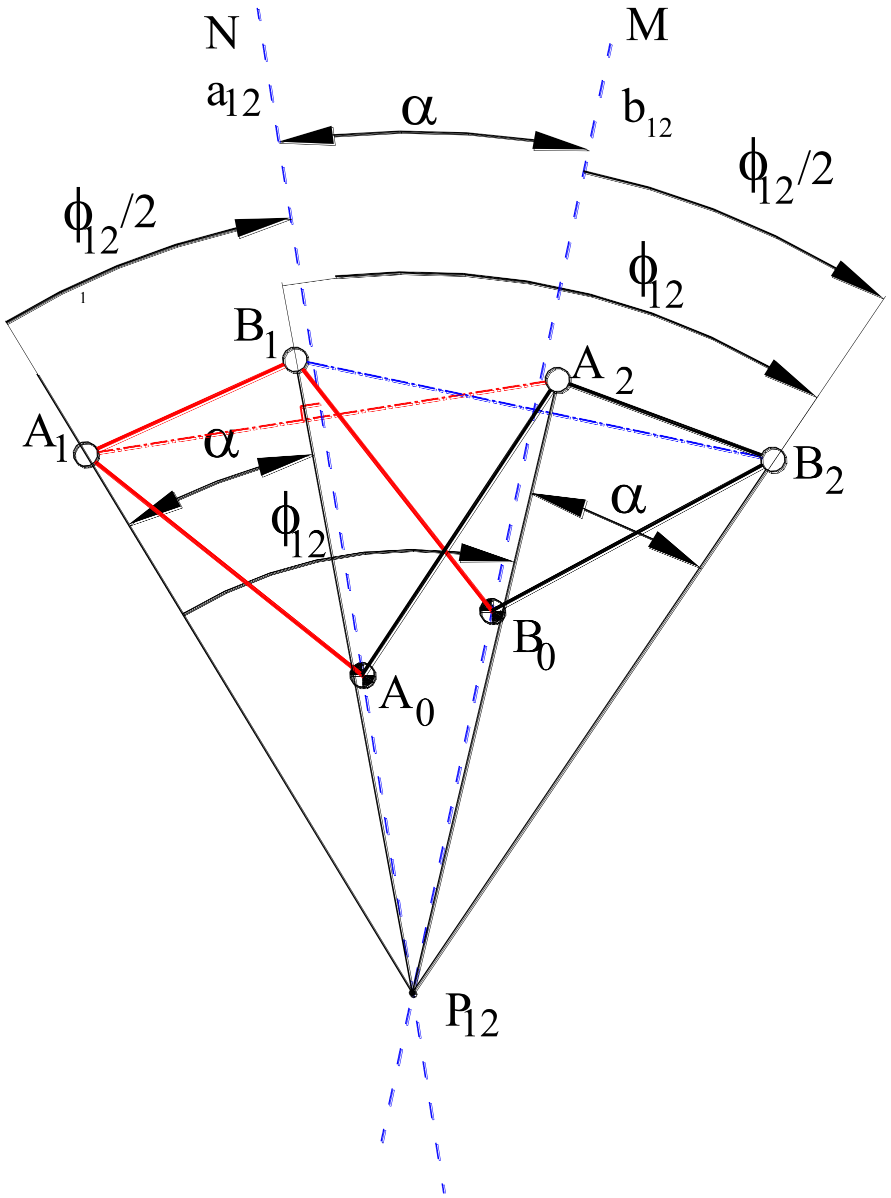
$\angle MP_{12}B_2 = ½ \angle B_1P_{12}B_2 = ½ \phi_{12}$

which result in

$\angle A_1P_{12}B_2 = \phi_{12} + \alpha = \phi_{12} + \angle N P_{12}M$

Hence:

$\angle N P_{12}M = \alpha$



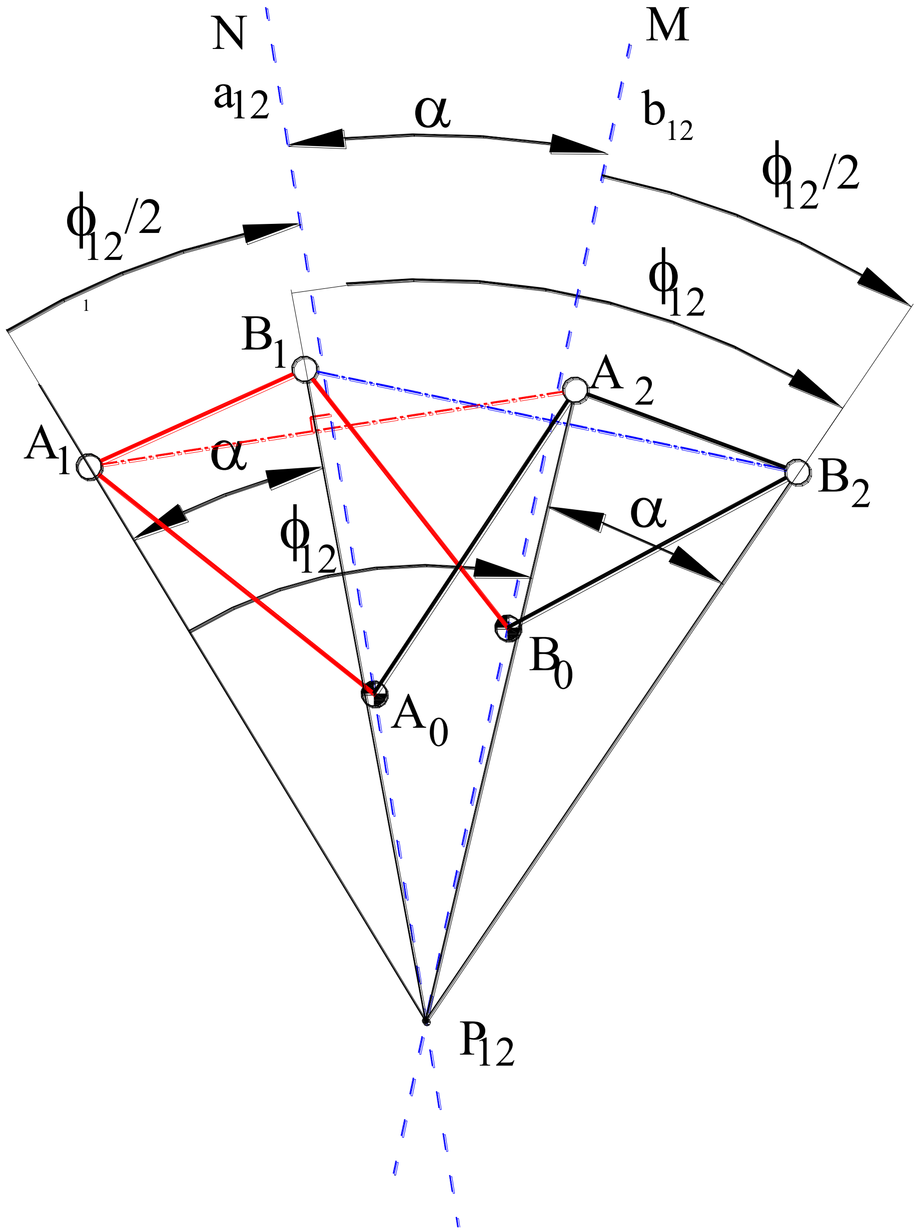
Some important relations:

1. The cranks A_0A and B_0B subtend the same angles or angles differing by π at the pole. This angle is equal to the half of the angle the rigid body makes at the pole. : $\frac{1}{2}\phi_{12}$ or $(\frac{1}{2}\phi_{12} + \pi)$. ϕ_{12} is the property of motion. It is the same for any point A and A_0 you select.

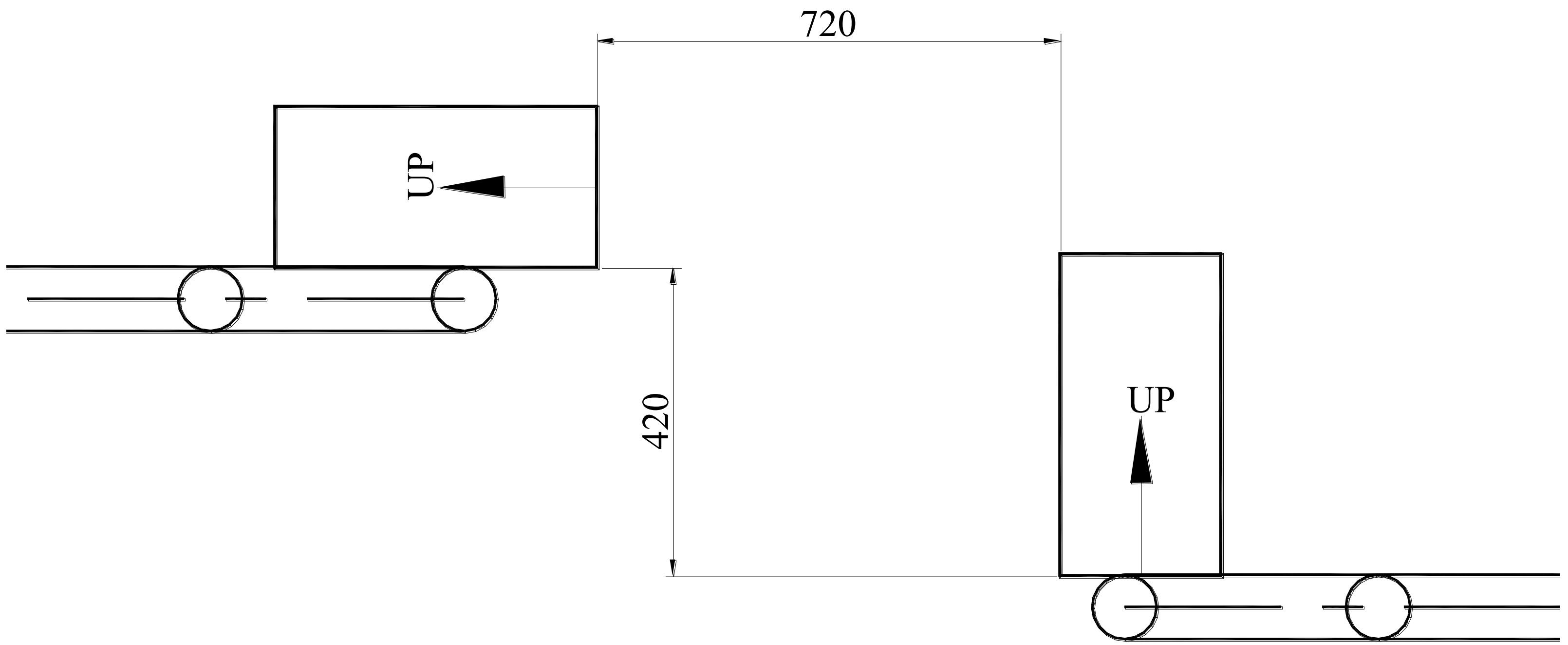
2. The coupler and the fixed link subtend equal angles or angles differing by π at the pole. This angle is α or $\alpha + \pi$. α is not the property of motion. (its value will change when you select different points A, B.

3. $P_{12}$and ϕ_{12} are the invariants of motion. Motion between two positions is completely defined by these two parameters.

P_{12} is the only point on the moving plane which is at the same location for the two positions 1 and 2.

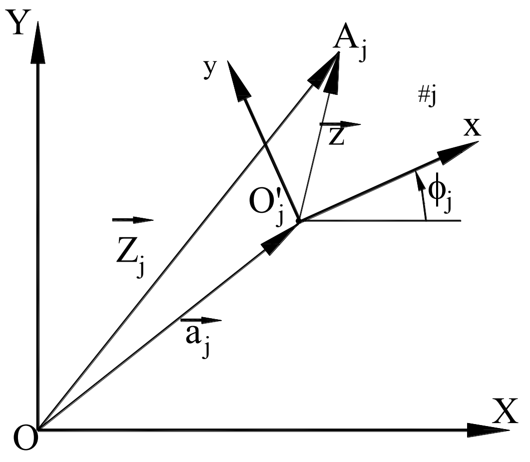


GeoGebra



Analytical Method

If point A does not move i.e. If $A_j = A_k$ then it is the pole P_{jk}:



$$Z_j = Z_k = P_{jk} \quad \text{and} \quad z = p_{jk}$$

$$a_j + p_{jk} e^{i\phi_j} = a_k + p_{jk} e^{i\phi_k}$$

or

$$p_{jk}(e^{i\phi_j} - e^{i\phi_k}) = a_k - a_j$$

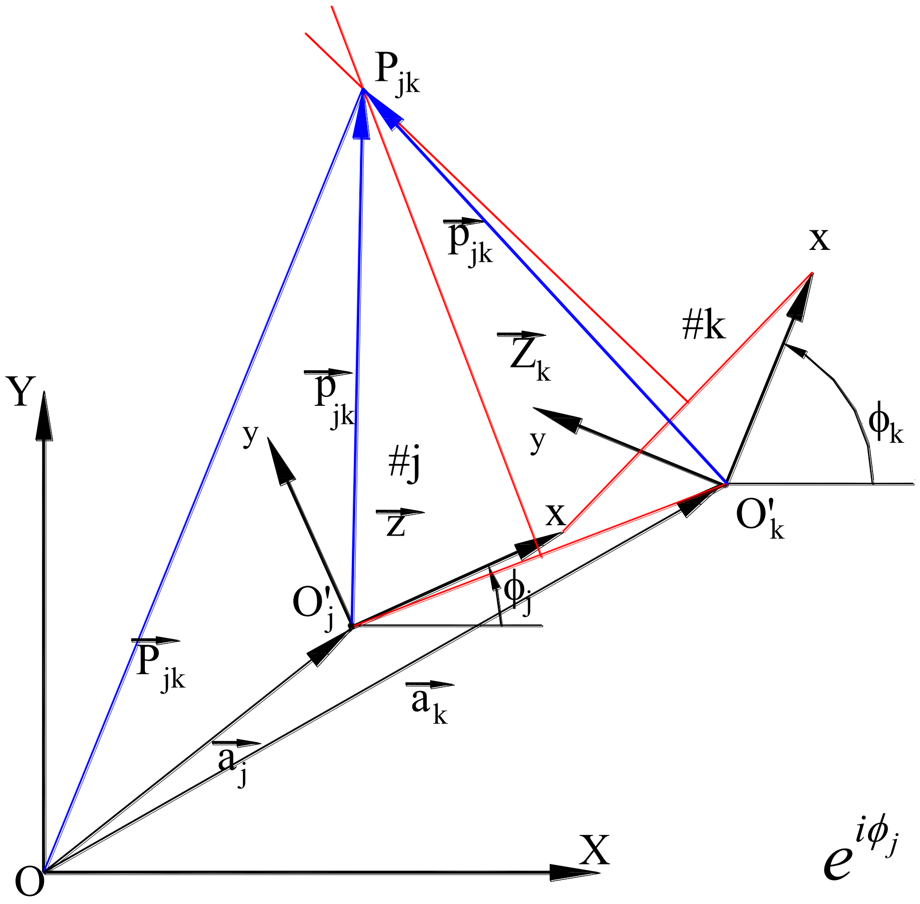
$$p_{jk} = \frac{a_k - a_j}{e^{i\phi_j} - e^{i\phi_k}}$$

Location of the pole with respect to moving frame

$$Z_j = a_j + z e^{i\phi_j}$$

$$Z_k = a_k + z e^{i\phi_k}$$

Pole



$$a_j + p_{jk} e^{i\phi_j} = a_k + p_{jk} e^{i\phi_k}$$

$$p_{jk}(e^{i\phi_j} - e^{i\phi_k}) = a_k - a_j$$

$$p_{jk} = \frac{a_k - a_j}{e^{i\phi_j} - e^{i\phi_k}}$$ = location of the pole w.r. to moving frame.

$$Z_j = Z_k = P_{jk} = a_j + p_{jk} e^{i\phi_j} = a_j + \left[\frac{a_k - a_j}{e^{i\phi_j} - e^{i\phi_k}}\right] e^{i\phi_j}$$

$$P_{jk} = \frac{a_j(e^{i\phi_j} - e^{i\phi_k}) + (a_k - a_j)e^{i\phi_j}}{e^{i\phi_j} - e^{i\phi_k}} = \frac{a_j e^{i\phi_j} - a_j e^{i\phi_k} + a_k e^{i\phi_j} - a_j e^{i\phi_j}}{e^{i\phi_j} - e^{i\phi_k}}$$

$$P_{jk} = \frac{a_k e^{i\phi_j} - a_j e^{i\phi_k}}{e^{i\phi_j} - e^{i\phi_k}}$$ = Location of the pole on the fixed frame.

$$e^{i\phi_j} = e_j \qquad e^{i\phi_k} = e_k$$

$Z_j = Z_k = P_{jk}$

$$p_{jk} = \frac{a_k - a_j}{e_j - e_k} \qquad P_{jk} = \frac{a_k e_j - a_j e_k}{e_j - e_j}$$

$z = p_{jk}$

$$P_{12}=\frac{a_2 e^{i\phi_1}-a_1 e^{i\phi_2}}{e^{i\phi_1}-e^{i\phi_2}}$$

```
Function Pole(A1, Fi1, A2, Fi2)
Dim Num(2), Denom(2), A(2), Del
Num(0) = A1(1) * Cos(Fi2) - A1(2) * Sin(Fi2) - (A2(1) * Cos(Fi1) - A2(2) * Sin(Fi1))    Real and imaginary parts of
Num(1) = A1(1) * Sin(Fi2) + A1(2) * Cos(Fi2) - (A2(1) * Sin(Fi1) + A2(2) * Cos(Fi1))    numerator
Denom(0) = Cos(Fi2) - Cos(Fi1)    Real and imaginary parts of denumerator
Denom(1) = Sin(Fi2) - Sin(Fi1)
Del = Denom(0) ^ 2 + Denom(1) ^ 2    Denumerator multiplied by its complex conjugate
A(0) = (Num(0) * Denom(0) + Num(1) * Denom(1)) / Del
A(1) = (Num(1) * Denom(0) - Num(0) * Denom(1)) / Del    Real and imaginary parts of the pole
Pole = A
End Function
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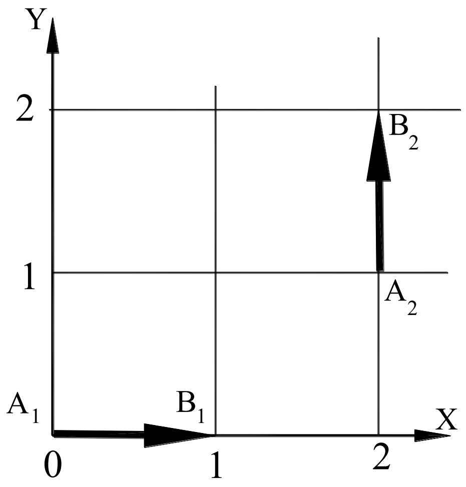
Example

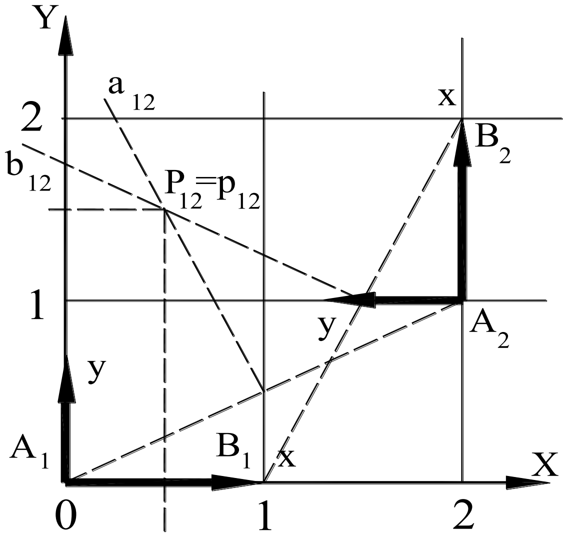


$$p_{12} = \frac{a_2 - a_1}{e^{i\phi_1} - e^{i\phi_2}} = \frac{2+1i-0}{1\text{-}i} = \frac{(1+i)(2+i)}{(1+i)(1\text{-}i)}$$

$$p_{12} = \frac{1+3i}{2} = \frac{1}{2} + \frac{3}{2}i$$ Location of the pole on the moving frame

$$P_{12} = \frac{a_2 e^{i\phi_1} - a_1 e^{i\phi_2}}{e^{i\phi_1} - e^{i\phi_2}} = \frac{(2+i)*1-0*i}{1\text{-}i} = \frac{2+i}{1-i}$$

$$P_{12} = \frac{1}{2} + \frac{3}{2}i = p_{12}$$ Location of the pole on the fixed frame



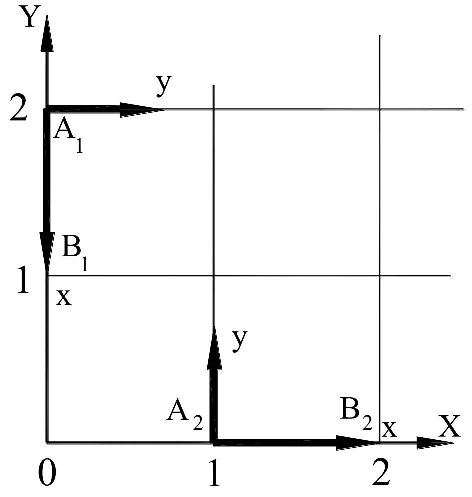
$\mathbf{a_1} = 0$, $\mathbf{a_2} = 2+1i$, $\phi_1 = 0^0$, $\phi_2 = \pi/2$.

$e^{i\phi_1} = e^{i0} = 1$ and $e^{i\phi_2} = e^{i\pi/2} = i$

$\mathbf{P}_{12} = \mathbf{p}_{12}$

Because the fixed frame and the first position of the moving frame are coincident!!!

Example



$$p_{12} = \frac{a_2 - a_1}{e^{i\phi_1} - e^{i\phi_2}} = \frac{1-2i}{-i-1} = -\frac{(1-i)(1-2i)}{(1+i)(1-i)}$$

$$p_{12} = \frac{1+3i}{2} = \frac{1}{2} + \frac{3}{2}i$$ Location of the pole on the moving frame

$$P_{12} = \frac{a_2 e^{i\phi_1} - a_1 e^{i\phi_2}}{e^{i\phi_1} - e^{i\phi_2}} = \frac{1*-i-2i*1}{-i-1} = \frac{3i}{1+i}$$

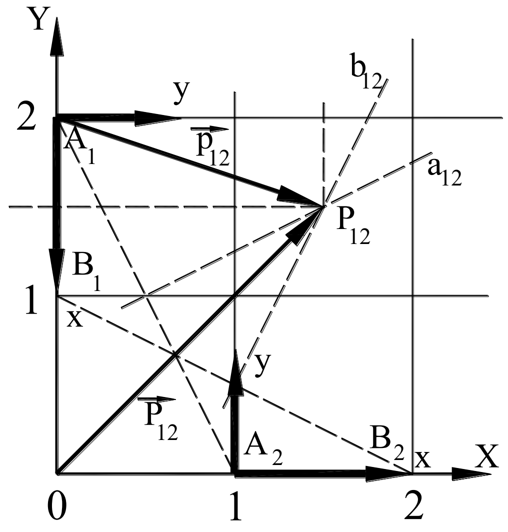
$$P_{12} = \frac{3}{2} + \frac{3}{2}i$$

Location of the pole on the fixed frame

$a_1 = 2i$, $a_2=1$, $\phi_1 = -\pi/2$, $\phi_2=0$

$e^{i\phi_1} = e^{-i\pi/2} = -i$ and $e^{i\phi_2} = e^{i0} = 1$

($\phi_{12}=\pi/2$ (=90))



$$Z_j = a_j + ze^{i\phi_j} = a_j + ze_j$$

$$Z_j - Z_k = a_j - a_k + z(\mathrm{e}_j - e_k)$$

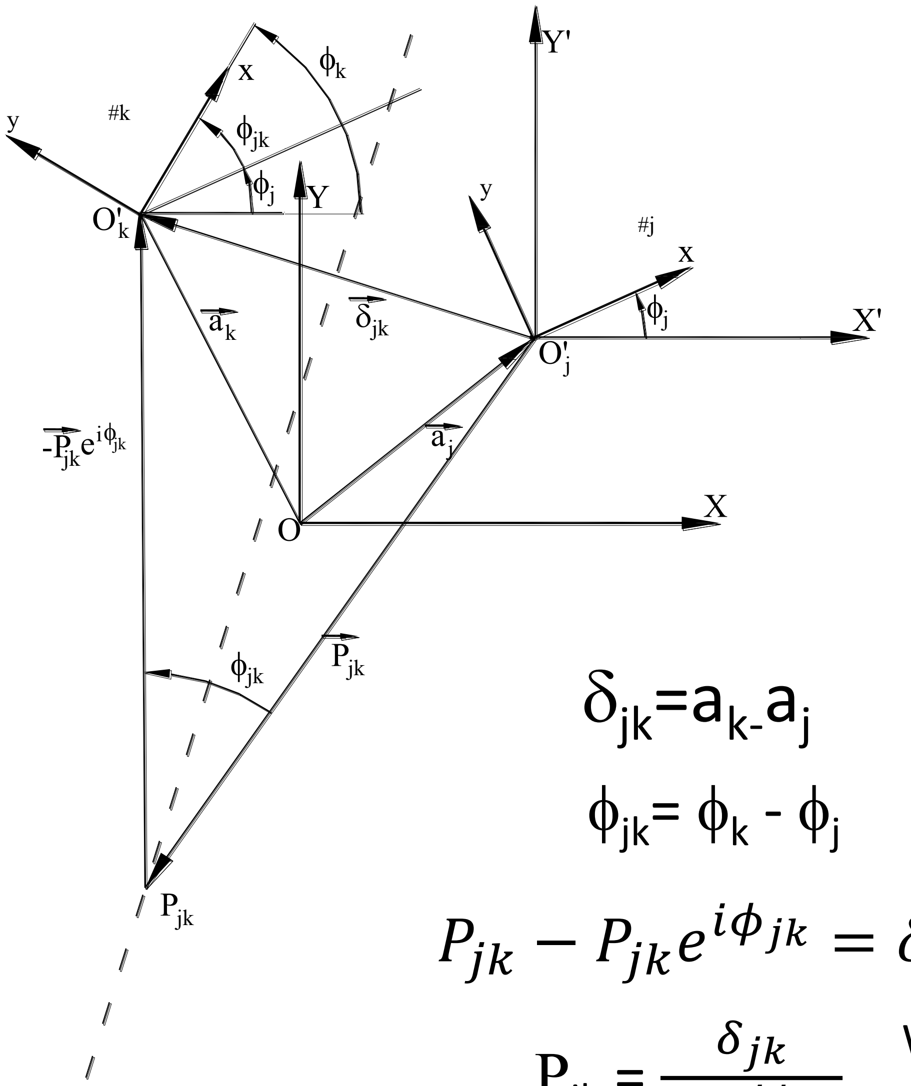
$$=(\mathrm{e}_j - e_k)\left[\frac{a_j - a_k}{(\mathrm{e}_j - e_k)} + z\right] = (\mathrm{e}_j - e_k)\left[z - p_{jk}\right]$$

$$Z_j - Z_k = (\mathrm{e}_j - e_k)\left[z - p_{jk}\right]$$

Taking the complex conjugate and noting: $e^{-i\theta} = \frac{1}{e^{i\theta}} \quad (\overline{\mathrm{e}} = \frac{1}{e})$

$$\overline{Z}_j - \overline{Z}_k = -\frac{(\mathrm{e}_j - e_k)}{\mathrm{e}_j\, e_k}\left[\overline{z} - \overline{p}_{jk}\right]$$

An Alternative



$$\delta_{jk}=a_k-a_j$$

$$\phi_{jk}=\phi_k-\phi_j$$

$$P_{jk}-P_{jk}e^{i\phi_{jk}}=\delta_{jk}$$

$$P_{jk}=\frac{\delta_{jk}}{1-e^{i\phi_{jk}}}$$

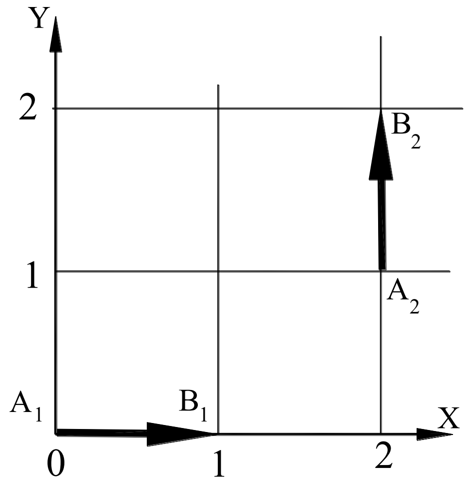
With respect to a fixed X'Y' axis, origin coincident with the jth position, paralel to XY, fixed coordinate frame.

Example

$\delta_{12}=2+i$, $\phi_{12}=\pi/2$

$$\vec{P}_{12}=\frac{2+i}{1-i}=\frac{1}{2}(2+i)(1+i)$$

$$\vec{P}_{12}=\frac{1}{2}+\frac{3}{2}i$$

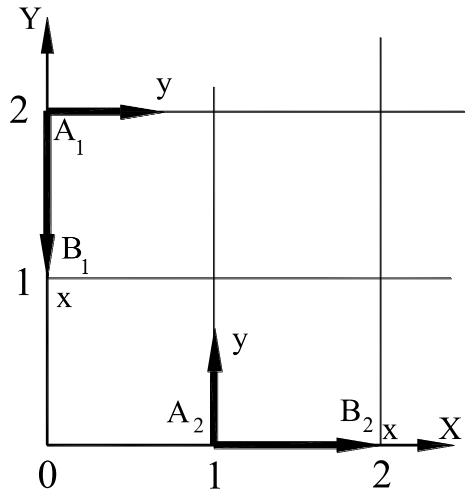


$a_1 = 2i$, $a_2=1$; $\boldsymbol{\delta_{12}}= \mathbf{a}_2 - \mathbf{a}_1$

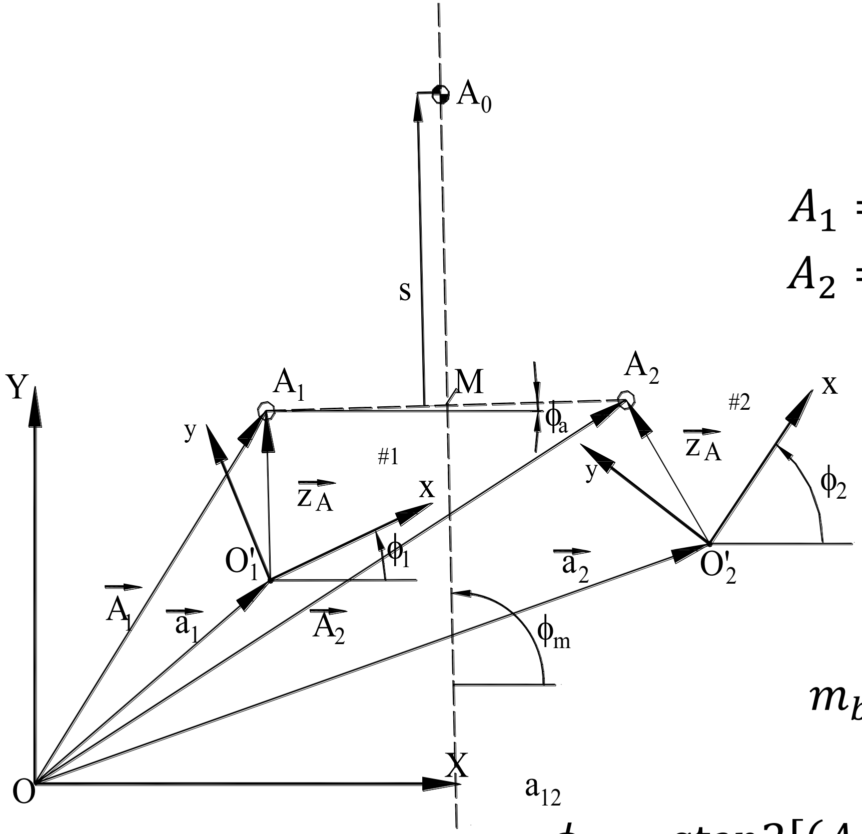
$\boldsymbol{\delta_{12}} =1-2i$: $\phi_{12}=\pi/2$

$$\vec{P}_{12}=\frac{1-2i}{1-i}=\frac{1}{2}(1-2i)(1+i)$$

$$\vec{P}_{12}=\frac{3}{2}-\frac{1}{2}i$$



How to determine centerpoint from circlepoint



a) Coordinates of point A on the fixed frame is given: z_A=x+iy is assumed given

b) If the coordinates of A_1 is given:

You specify 2 parameters

$$A_1 = a_1 + z_A e^{i\phi_1}$$
$$A_2 = a_2 + z_A e^{i\phi 2}$$

$$z_A = (A_1 - a_1)e^{-i\phi_1}$$
$$A_2 = a_2 + z_A e^{i\phi_2}$$

$$A_M = (A_1 + A_2)/2$$

$A_{xM}=(A_{1x}+A_{2x})/2$, $A_{YM}=(A_{1y}+A_{2Y})/2$

$$m_a = \frac{A_{2Y} - A_{1Y}}{A_{2X} - A_{1X}} \quad \textit{slope of } A_1A_2$$

$$m_b = -\frac{A_{2X} - A_{1X}}{A_{2Y} - A_{1Y}} \quad \textit{slope of perpendicular bisector line}$$

$$\phi_a = \mathrm{atan2}[(A_{2X} - A_{1X}); (A_{2X} - A_{1X})] \quad \textit{Angle } A_1A_2 \textit{ makes with X axis}$$
$$\phi_b = \phi_a + \pi/2$$

Specify s, $X_{A0} = (A_{1X}+A_{2X})/2 + s\cos(\phi_b)$; $Y_{A0} = (A_{1Y}+A_{2Y})/2 + s\sin(\phi_b)$

Repeat for B

$A_0 = (X_{A0}, Y_{A0})$ or $Z_{A0} = X_{A0} + iY_{A0}$

If the coordinates of A_1 is given:

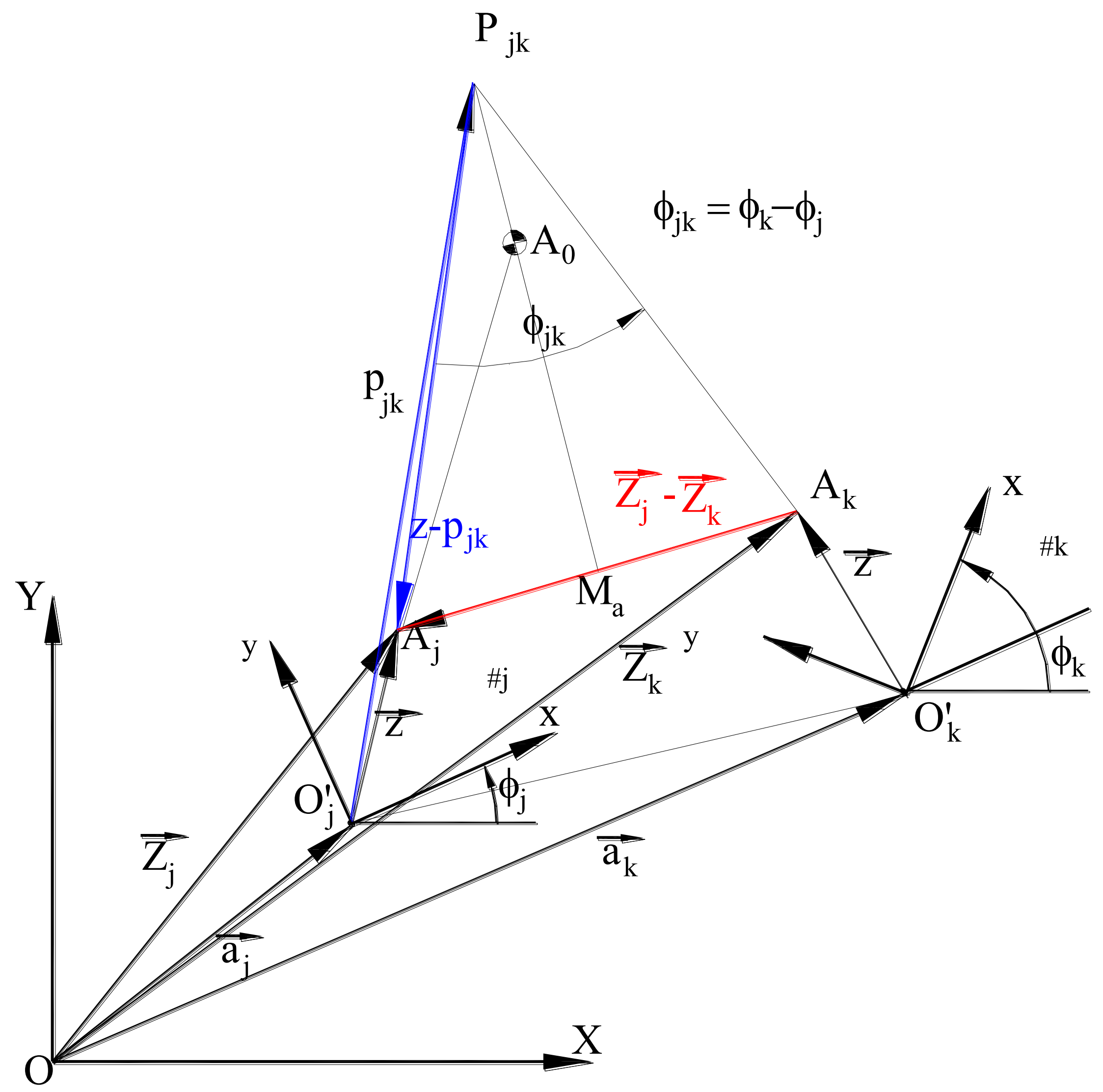
First determine P_{12}. Determine the vector $P_{12}A_1$ and rotate this vectorabout P_{12} by angle $\phi_{12}=\phi_2-\phi_1$ to determine A_2.

Next Determine the midpoint M and the angle ϕ_a (angle made by A_1A_2 w.r to X axis. $\phi_b=\phi_a+\pi/2$ is the angle made by the perpendicular bisector w.r. To X axis.

Determine the coordinates of A_0

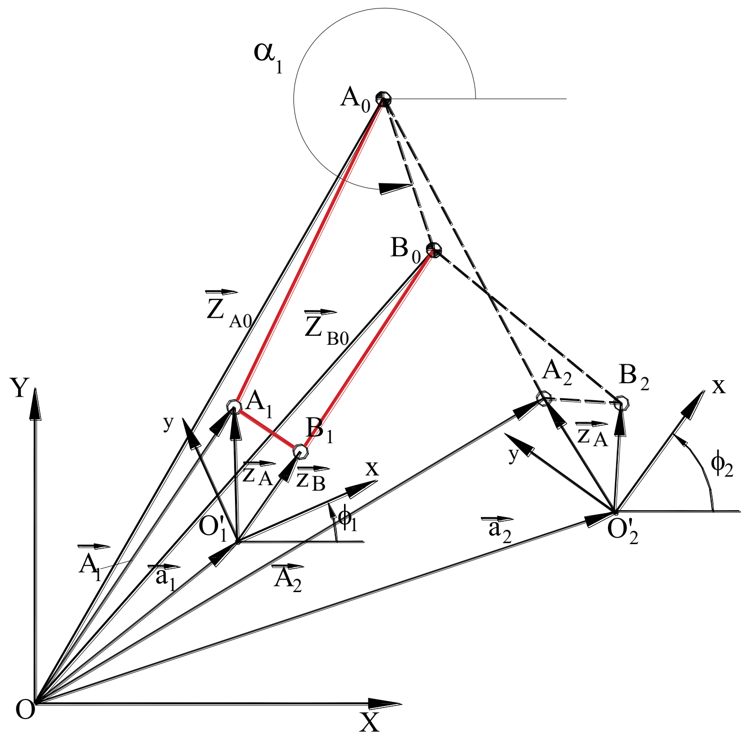
$$X_{A0} = (A_{1X}+A_{2X})/2 + s\cos(\phi_b)\ ;\ Y_{A0} = (A_{1Y}+A_{2Y})/2 + s\sin(\phi_b)$$

$$A_0= (X_{A0},Y_{A0}) \text{ or } Z_{A0}=X_{A0} + iY_{A0}$$



How to analyze?

a) Determine the link lengths
b) Determine the angular position of the fixed link
c) Decide on the input link and determine its angular rotation in between the two positions.



From the synthesis you obtain 4 vectors which show the coordinates of the moving and fixed pivots:

A_1, B_1, Z_{A0}, Z_{B0}

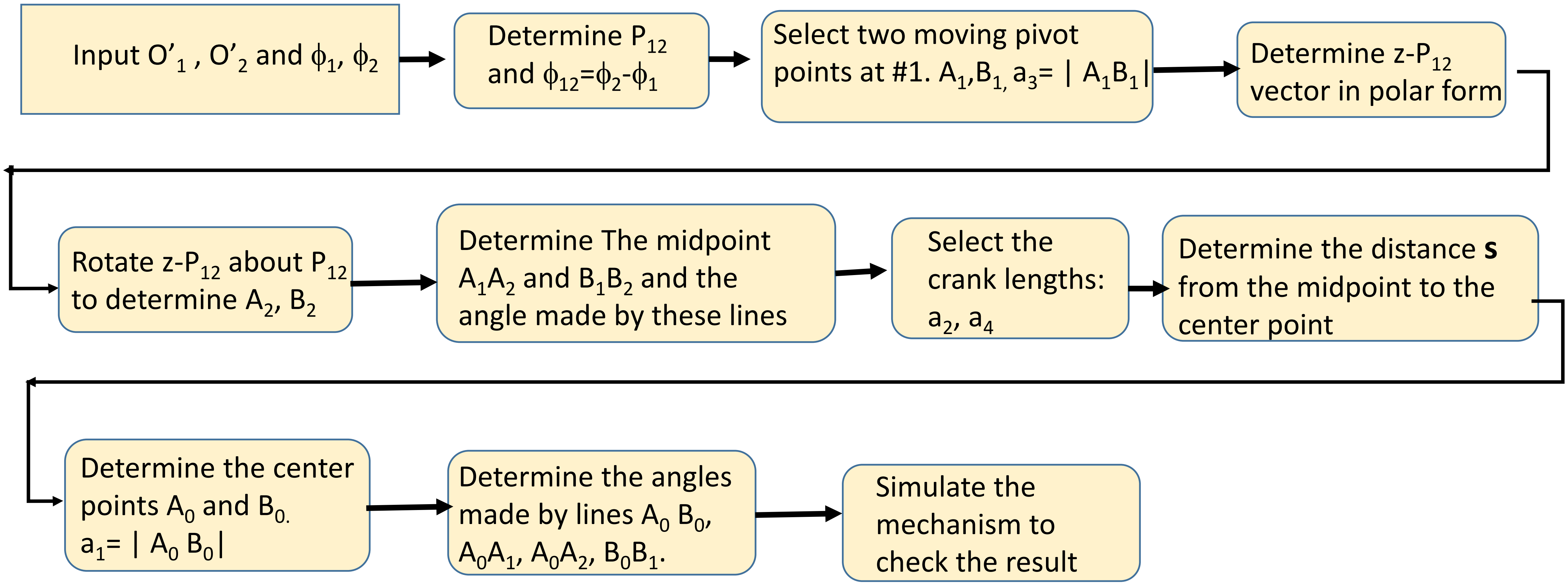
$\mathbf{A_0A_1}=\mathbf{A_1}-\mathbf{Z_{A0}}$. $\quad a_2=|\mathbf{A_1}-\mathbf{Z_{A0}}|$.

$\mathbf{B_0B_1}=\mathbf{B_1}-\mathbf{Z_{B0}}$ $\quad a_4=|\mathbf{B_1}-\mathbf{Z_{B0}}|$.

$\mathbf{A_1B_1}=\mathbf{B_1}-\mathbf{A_1}$, $\mathbf{A_0B_0}=\mathbf{Z_{B0}}-\mathbf{Z_{A0}}$ and $a_1=|\mathbf{Z_{B0}}-\mathbf{Z_{A0}}|$, $\quad a_3=|\mathbf{B_1}-\mathbf{A_1}|$

Angular positions are determined from the arguments of these vectors. i.e

$$\alpha_1 = \arg(Z_{B0} - Z_{A0}) = a\tan 2((A_{BoX} - A_{A0X});(A_{BoY} - A_{A0y}))$$


Input O'_1 , O'_2 and ϕ_1, ϕ_2
Determine P_{12} and $\phi_{12}=\phi_2-\phi_1$
Select two moving pivot points at #1. A_1,B_1, a_3= | A_1B_1|
Determine z-P_{12} vector in polar form
Rotate z-P_{12} about P_{12} to determine A_2, B_2
Determine The midpoint A_1A_2 and B_1B_2 and the angle made by these lines
Select the crank lengths: a_2, a_4
Determine the distance **s** from the midpoint to the center point
Determine the center points A_0 and B_0. a_1= | A_0 B_0|
Determine the angles made by lines A_0 B_0, A_0A_1, A_0A_2, B_0B_1.
Simulate the mechanism to check the result